

Complexity of F4 tracer for Gröbner bases computations

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Gröbner bases

Polynomials $\mathcal{G} = (g_1, \dots, g_\ell)$ in $\mathbb{K}[x_1, \dots, x_n]$, monomial order \succ
 \mathcal{G} is a Gröbner basis of $(\langle \mathcal{G} \rangle, \succ)$ if

$$\langle \text{lm}_\succ(g_1), \dots, \text{lm}_\succ(g_\ell) \rangle = \langle \text{lm}_\succ(\langle \mathcal{G} \rangle) \rangle$$

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Examples:

- 1 Lexicographic ordering
- 2 Graded reverse lexicographic ordering (grevlex)

Lexicographic Gröbner basis

$$I = \langle -2xy - 2x - 2y^2 - y + 3, x^2 + 3x - 2y^2 - 3y \rangle$$

$$\mathcal{G}_{\text{lex}} = \{x - y^3 - y^2 - y, y^4 + 2y^3 + 3y^2 - 2y + 2\}$$

Applications: Decide whether $I = J$, decide whether $f \in I$, compute $I \cap J$

(f_1, \dots, f_n) **generic** polynomials in $\mathbb{K}[x_1, \dots, x_n]$ with $\deg(f_i) = \delta$:

$$\begin{cases} f_1(x_1, \dots, x_n) = 0 \\ \vdots \\ f_n(x_1, \dots, x_n) = 0 \end{cases}$$

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polynomial system $\xrightarrow[\text{[Buchberger]}]{\text{[F4, F5]}}$ grevlex Gröbner basis $\xrightarrow[\text{[FGLM 1993]}]{\text{change of ordering}}$ lex Gröbner basis

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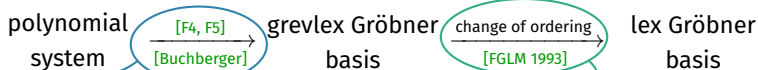
State of the art: ($t \leq \deg(I)$)

- [FGLM 1993] $\mathcal{O}(n \deg(I)^3)$
- [Faugère, Mou 2017] $\tilde{\mathcal{O}}(t \deg(I)^2)$
- [Neiger, Schost 2020] $\tilde{\mathcal{O}}(n \deg(I)^\omega)$
- [Berthomieu, Neiger, Safey El Din 2022] $\tilde{\mathcal{O}}(t^{\omega-1} \deg(I))$

$2 \leq \omega \leq 3$: exponent of matrix multiplication

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State of the art:

- Naive complexity bound $\mathcal{O}\left(\binom{n+D}{D}^\omega\right)$

- Analysis on F5-POT

[Bardet, Faugère, Salvy 2015]

- [Berthomieu, Neiger, Safey El Din 2022] $\tilde{\mathcal{O}}(t^{\omega-1} \deg(I))$

Goal: refined complexity analysis for F4 tracer

$2 \leq \omega \leq 3$: exponent of matrix multiplication

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[Buchberger 1965]

critical pairs



[Faugère 1999](F4)

[Faugère 2002](F5)

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linear algebra



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• **S-polynomial** [Buchberger 1965]

For polynomials f_1, f_2 ,

$$S(f_1, f_2) = \frac{\text{LCM}(\text{lm}_{>}(f_1), \text{lm}_{>}(f_2))}{\text{lm}_{>}(f_1)} f_1 - \frac{\text{LCM}(\text{lm}_{>}(f_1), \text{lm}_{>}(f_2))}{\text{lm}_{>}(f_2)} f_2$$

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• **Macaulay Matrices** [Macaulay 1902]

$$f = 3x_1^2 - 7x_2^2 + \dots \longrightarrow \begin{pmatrix} x_1^2 & x_1x_2 & x_2^2 & x_1x_3 & x_2x_3 & x_3^2 & x_1 & x_2 & x_3 & 1 \\ 3 & 0 & -7 & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}$$

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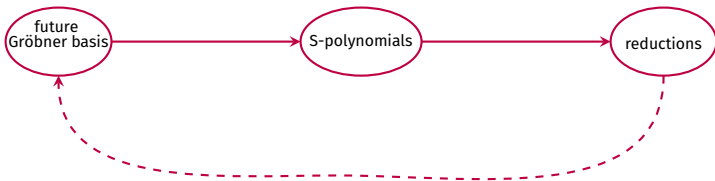
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In Software [msolve 2021]: F4 is traced [Traverso 1989]

- 1 F4 over $\mathbb{Q} \implies$ F4 over $\mathbb{Z}/p_i\mathbb{Z}$ for all $(p_i)_{i \in \{1, \dots, r\}}$ prime integers
- 2 For p_2, \dots, p_r , re-use information from the computation for p_1

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Upper triangular

Reducers

S-polynomials

$$\left\{ \begin{array}{c} \text{Upper triangular} \\ \text{Reducers} \end{array} \right\} \left(\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right) \implies \left(\begin{array}{c|c} Id & A^{-1}B \\ \hline 0 & M \end{array} \right)$$

Note: In the original image, the matrix $\begin{pmatrix} A & B \\ \hline C & D \end{pmatrix}$ is annotated with a red circle around A , a red bracket on the left, and a blue bracket on the left.

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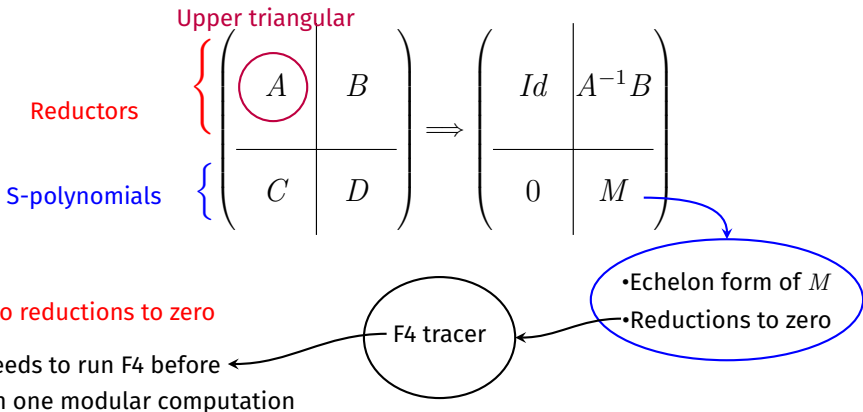
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•Echelon form of M
•Reductions to zero

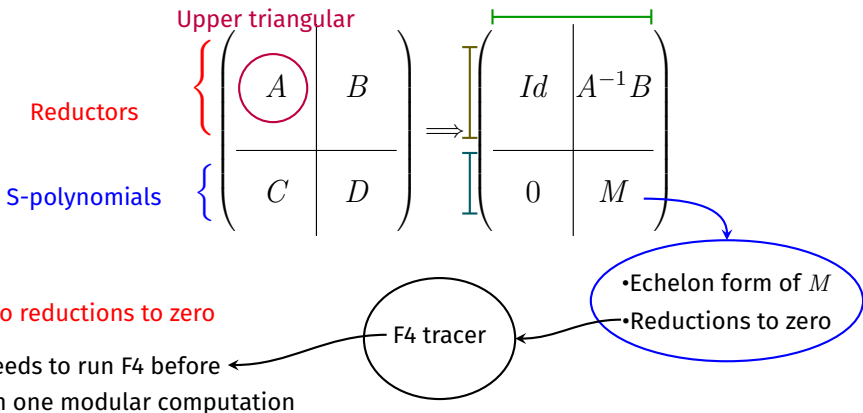
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Complexity: $\mathcal{O}(\#(\text{new GB elements}) \times \#(\text{reducers}) \times \#(\text{support}))$

Examples	System data		msolve single modular computation		
	degree	Variables	F4 (p_1)	F4 Tracer (p_{i_0} for $i_0 \neq 1$)	Sparse FGLM
Katsura-11	2	10	6.26	0.22	0.17
Katsura-12	2	11	22.09	1.25	1.02
Katsura-13	2	12	178.68	8.38	13.52
Katsura-14	2	13	1366	53.42	81.48
Katsura-15	2	14	9204	324	502

F4 tracer better than FGLM

FGLM polynomial in the degree of the ideal

Practical results do not match with complexities

Why is F4 Tracer this good?

For generic polynomials (f_1, \dots, f_n) of degree δ in $\mathbb{K}[x_1, \dots, x_n]$

Naive complexity

$$\mathcal{O}\left(\binom{n+D}{D}^\omega\right)$$

Macaulay bound [Lazard 1983]

$$D = n(\delta - 1) + 1$$

New complexity of F4 tracer on a top reduction context

$$\mathcal{O}\left(\sum_{d=\lfloor D/2 \rfloor + 1}^D \binom{n+d}{d} (B_{d,n})^{\omega-2} \sum_{\substack{0 \leq j \leq 2(n-1)+1 \\ D-d+j \leq d}} \binom{n-2+D-d+j}{D-d+j}\right)$$

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support
still naive

new GB elements
exact formula

reducers
new bound

$\omega = 3$	$n = 5$	$n = 10$	$n = 15$	$n = 20$	$n = 25$	$n = 30$	$n = 35$
F4 Tracer $\delta = 5$	$\delta^{2.71n}$	$\delta^{3.13n}$	$\delta^{3.28n}$	$\delta^{3.40n}$	$\delta^{3.44n}$	$\delta^{3.47n}$	$\delta^{3.49n}$
Binomial $\delta = 5$	$\delta^{4.13n}$	$\delta^{4.33n}$	$\delta^{4.42n}$	$\delta^{4.46n}$	$\delta^{4.49n}$	$\delta^{4.52n}$	$\delta^{4.53n}$
F4 Tracer $\delta = 10$	$\delta^{2.51n}$	$\delta^{2.92n}$	$\delta^{3.07n}$	$\delta^{3.14n}$	$\delta^{3.18n}$	$\delta^{3.22n}$	$\delta^{3.26n}$
Binomial $\delta = 10$	$\delta^{3.82n}$	$\delta^{3.98n}$	$\delta^{4.05n}$	$\delta^{4.08n}$	$\delta^{4.11n}$	$\delta^{4.12n}$	$\delta^{4.14n}$

Hypotheses:

- 1 Moreno-Socías conjecture on the grevlex monomial staircase
[Moreno-Socías 2003] → **Stability** (generic hypothesis)
- 2 Dense polynomials, regularity and semi-regularity on some truncations of the polynomials
- 3 Polynomials have the same degree

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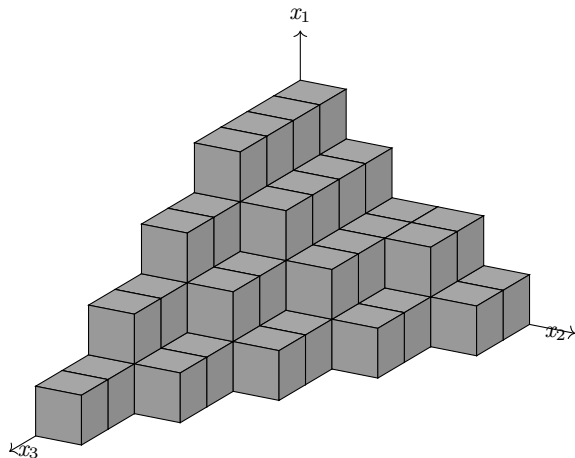
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Buchberger criterion: [Buchberger 1979] f_1, f_2, f_3 polynomials:

$$\boxed{\text{lm}_{\succ}(f_1) \mid \text{LCM}(\text{lm}_{\succ}(f_2), \text{lm}_{\succ}(f_3))} \quad \Longrightarrow \quad \boxed{(f_1, f_2) \text{ and } (f_1, f_3) \text{ absorb } (f_2, f_3)}$$

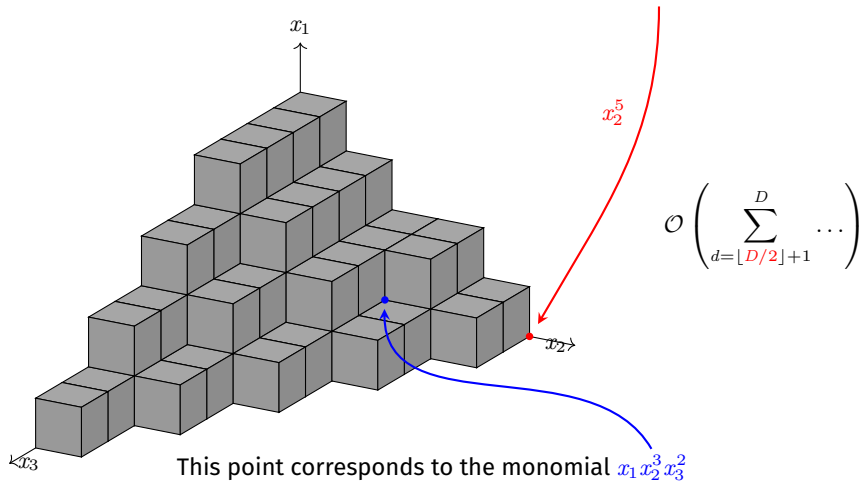
Grevlex monomial staircase

Let $f_1, f_2, f_3 \in \mathbb{K}[x_1, x_2, x_3]$ and $\deg(f_i) = 4 \rightarrow D = 10$



Grevlex monomial staircase

Let $f_1, f_2, f_3 \in \mathbb{K}[x_1, x_2, x_3]$ and $\deg(f_i) = 4 \rightarrow D = 10 \rightarrow D/2 = 5$



Closed in the first direction (x_1)

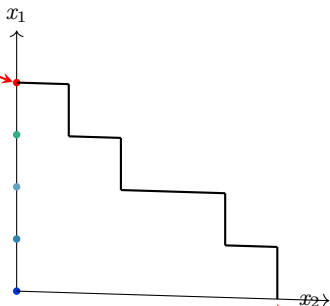


How the $B_{d,n}$ are defined?

Lemma: For all $\lfloor D/2 \rfloor + 1 \leq d \leq D$, the number of new Gröbner basis elements of degree d is generically $B_{d,n}$ with $\left[\frac{(1-t^\delta)^n}{(1-t)^{n-1}} \right] = \sum_{k=0}^{+\infty} B_{D-k,n} t^k$.

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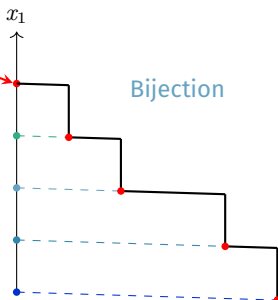
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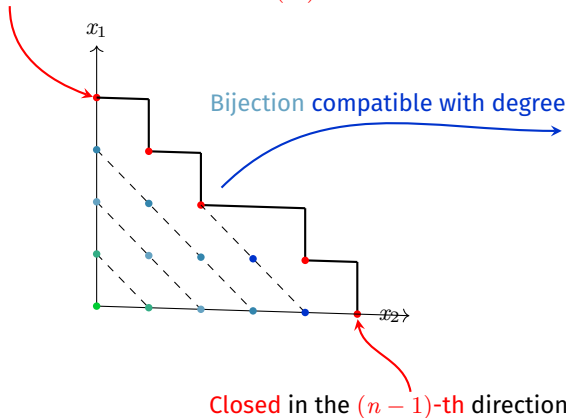
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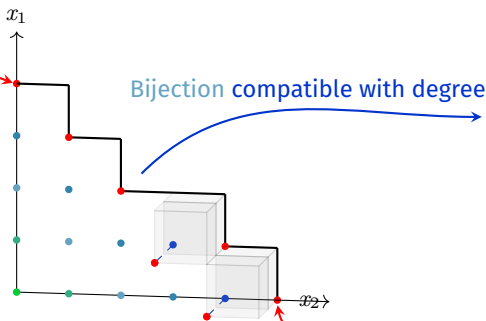


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$$B_{6,3} = 2$$



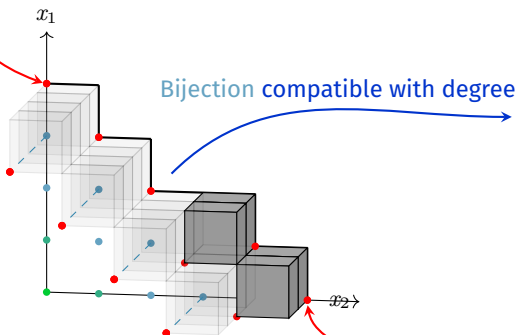
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$$B_{6,3} = 2$$

$$B_{7,3} = 4$$

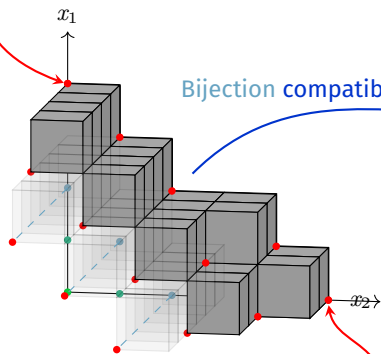
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Bijection compatible with degree

$$B_{6,3} = 2$$

$$B_{7,3} = 4$$

$$B_{8,3} = 3$$

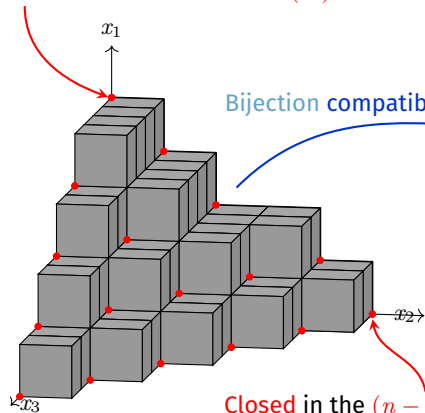
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$$B_{8,3} = 3$$

$$B_{9,3} = 2$$

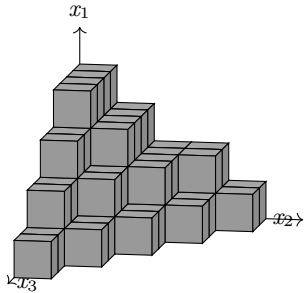
$$B_{10,3} = 1$$

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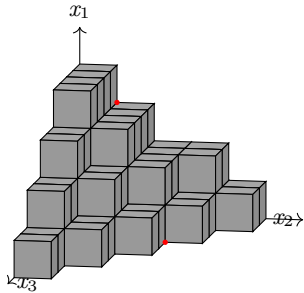
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How do we enumerate the reducers?

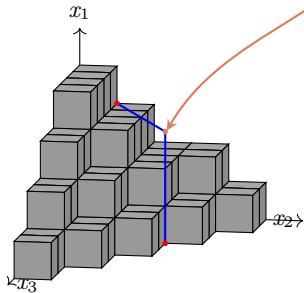


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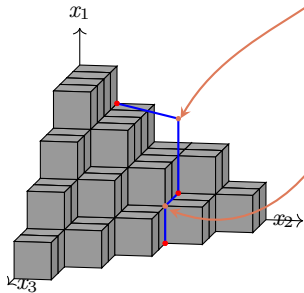
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leading monomial of the pair



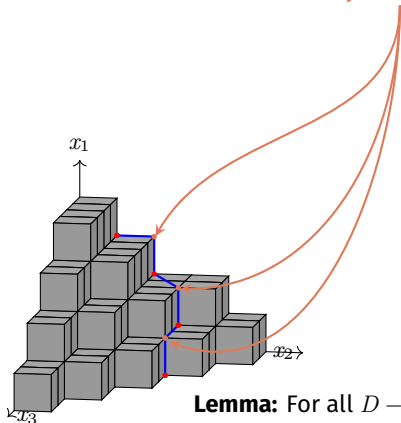
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Lemma: For all $D - \lfloor D/2 \rfloor - 1 \geq d \geq 0$ and (f_1, f_2) a critical pair considered by F4 tracer and $\deg(\text{LCM}(\text{lm}_{\succ}(f_1), \text{lm}_{\succ}(f_2))) = m$.

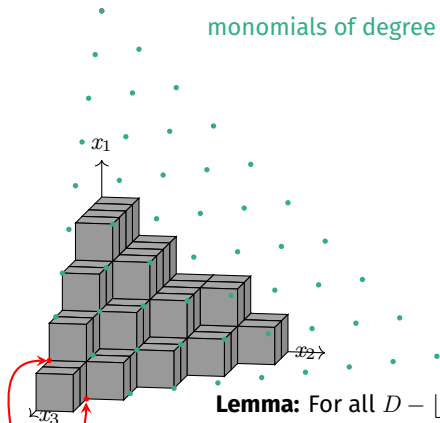
If $\deg(m) = D - d$, then generically:

$$D - 2d - 2(n - 1) - 1 \leq \deg_{x_n}(m) \leq D - 2d$$

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monomials of degree 9

$$\binom{n-1+d}{d}$$



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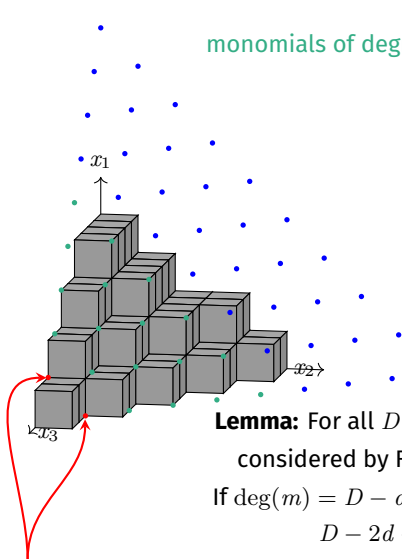
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new leading monomials at degree 9

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monomials of degree 9



$$\binom{n-1+d}{d}$$

$$\sum_{\substack{0 \leq j \leq 2(n-1)+1 \\ D-d+j \leq d}} \binom{n-2+D-d+j}{D-d+j}$$

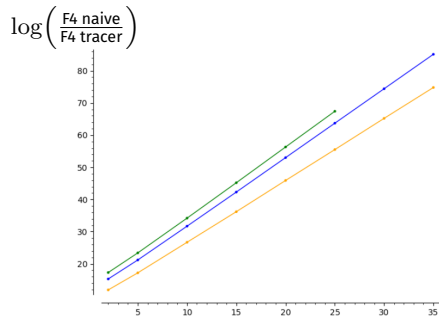
• Useless monomials

Lemma: For all $D - \lfloor D/2 \rfloor - 1 \geq d \geq 0$ and (f_1, f_2) a critical pair considered by F4 tracer and $\deg(\text{LCM}(\text{lm}_>(f_1), \text{lm}_>(f_2))) = m$.

If $\deg(m) = D - d$, then generically:

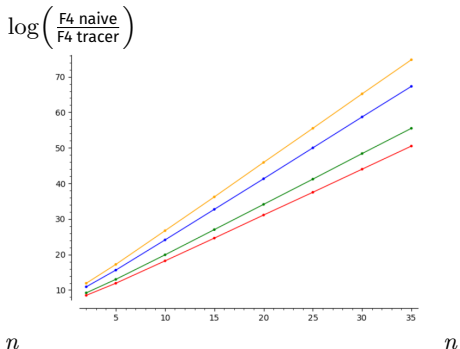
$$D - 2d - 2(n - 1) - 1 \leq \deg_{x_n}(m) \leq D - 2d$$

new leading monomials at degree 9



fixed $\omega = 3$

$\delta = 10$ $\delta = 20$ $\delta = 30$

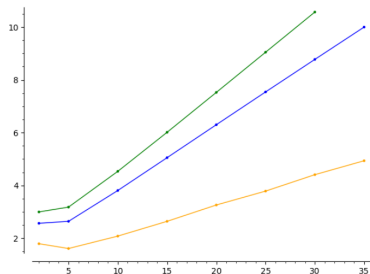


fixed $\delta = 10$

$\omega = 3$ $\omega = 2.8$ $\omega = 2.5$ $\omega = 2.37$

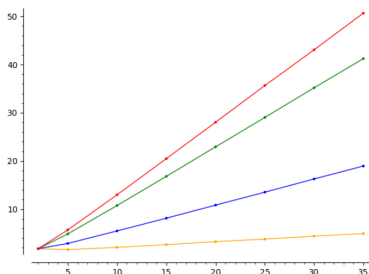
Naive complexity bound

$$\mathcal{O}\left(\binom{n+D}{D}^\omega\right)$$

$\log\left(\frac{\text{F5-POT}}{\text{F4 tracer}}\right)$


fixed $\omega = 3$

$\delta = 10$ $\delta = 20$ $\delta = 30$

 $\log\left(\frac{\text{F5-POT}}{\text{F4 tracer}}\right)$


n

fixed $\delta = 10$

$\omega = 3$ $\omega = 2.8$ $\omega = 2.5$ $\omega = 2.37$

n

Complexity of F5-POT

[Bardet, Faugère, Salvy 2015]

$$\mathcal{O}\left(\sum_{d=\delta}^D \binom{n+d}{d} \sum_{i=1}^n b_d^{(i)} \binom{i+d-1}{d}\right)$$

Conclusion & Perspectives

Conclusion:

- ✓ We obtained a new **bound** on the **number of reducers in F4 tracer** in a **top reduction context**.
- ✓ We obtained a new **complexity bound on F4 tracer** in a **top reduction context** which gives an **exponential gain** with regard to the naive bound.
- ✓ The constant ω seems to give **F4 tracer** an **exponential gain** with regard to F5 when n goes to infinity.

Perspectives:

- 💡 A better understanding of grevlex Gröbner bases.
- 💡 Improve the bound on reducers.
- 💡 Work on a **full reduction context** to decrease the support of the rows.
- 💡 Adapt this work to the **F4 algorithm**.
- 💡 Localize where these algorithms can possibly be improved.