Gröbner basis computation

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Algebraic Numbers and Polynomial Systems

$$\begin{cases} x^2 + y^2 &= 5 \\ x + y &= 1 \end{cases} \implies 2y^2 - 2y - 4 = 0$$

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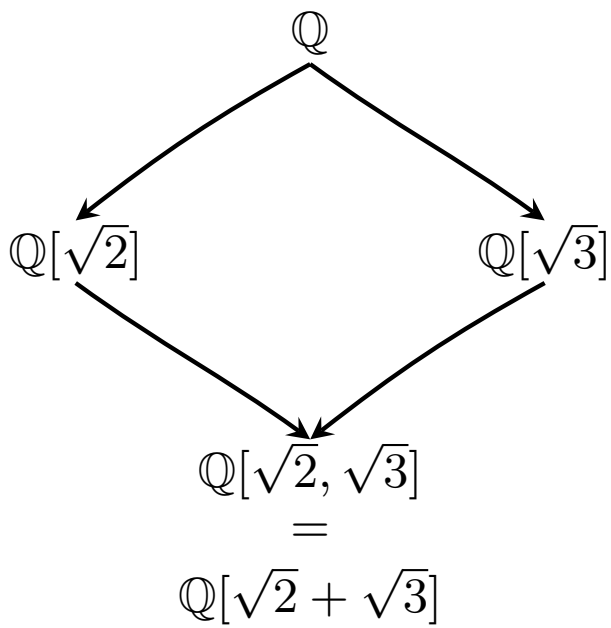

Primitive Element Theorem

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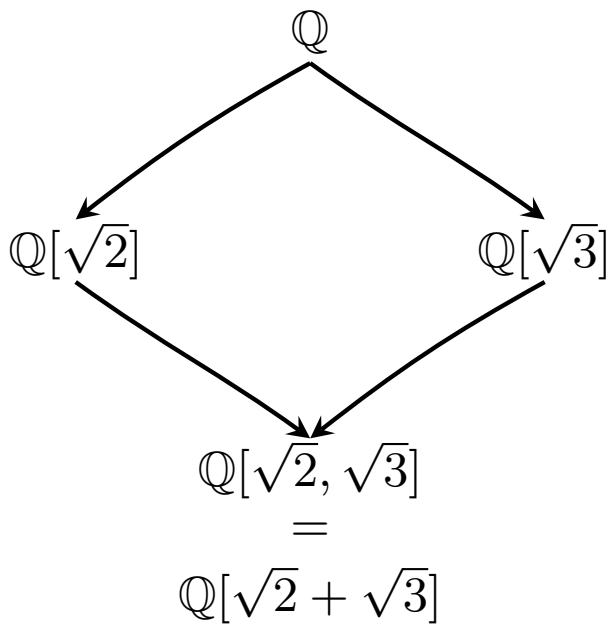

$$\begin{cases} x_1^2 - 2 &= 0 \\ x_2^2 - 3 &= 0 \\ x_3 - (x_1 + x_2) &= 0 \end{cases}$$

Primitive Element Theorem

The Legacies of Sylvester and Macaulay

$f = \sum_{k=0}^{d} a_k X^k$, $g = \sum_{k=0}^{e} b_k X^k$ in $\mathcal{R}[X]$

$$\begin{pmatrix} a_d & \cdots & \cdots & a_0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & a_d & \cdots & \cdots & a_0 \\ b_e & \cdots & \cdots & b_0 & 0 & \cdots & 0 \\ 0 & \ddots & & & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & & & \ddots & 0 \\ 0 & \cdots & 0 & b_e & \cdots & \cdots & b_0 \end{pmatrix}$$

$$\begin{array}{c} \mathcal{R}[X]_{<e} \oplus \mathcal{R}[X]_{<d} \longrightarrow \mathcal{R}[X]_{<d+e} \\ (U, V) \longmapsto UF + VG \end{array}$$

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b_e & \cdots & \cdots & b_0 & 0 & \cdots & 0 \\
0 & \ddots & & & \ddots & \ddots & \vdots \\
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$\mathsf{GCD}(f, g)$

$$f = qg + r$$

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bivariate case is doable using resultant

Why Solve Polynomial Systems in 2026?

Biology: Lotka–Volterra Predation Equations

$$\begin{cases} \dfrac{dx(t)}{dt} = \alpha x(t) - \beta x(t) y(t), \\ \dfrac{dy(t)}{dt} = \delta x(t) y(t) - \gamma y(t). \end{cases}$$

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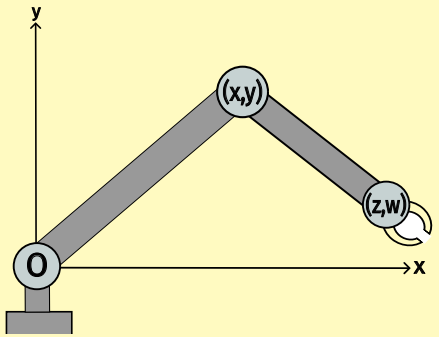

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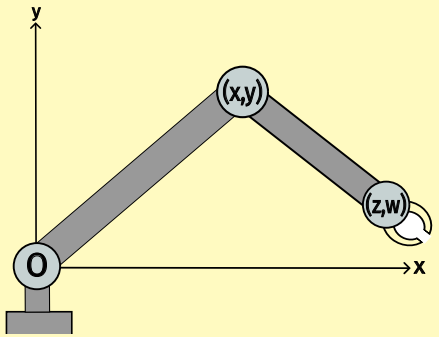

Application to **cryptography**

public key: a polynomial system $(f_1, \ldots, f_\ell)$

private key : a solution to the system $\zeta \in \mathbb{K}^n$ s.t. $f_1(\zeta) = \cdots = f_\ell(\zeta) = 0$

$$\text{message} \xrightarrow{\text{encryption}} \text{message} + \text{combination of the } f_i$$

$$\text{ciphertext} \xrightarrow{\text{decryption}} \text{ciphertext}(x = \zeta) = \text{message}$$

Some Foundational Problems and Historical Solutions

Ideal membership and elimination problem:

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The Weak Hilbert Nullstellensatz

Let $\mathbb{K}$ be an algebraically closed field and let $I \subseteq \mathbb{K}[x_1, \ldots, x_n]$ be an ideal which associated variety is empty. Then $I = \mathbb{K}[x_1, \ldots, x_n]$.

Knowing if the varity is empty is knowing if 1 lies in I or not.

Some Quantitative Bounds

Ideal membership:

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Existence of a solution:

NP-Hard problem

[Cook-Levin Theorem] and [Michael, Garey, 1979]

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- Hard problem for cryptography
- Lots of applications

From Euclid to Gröbner Bases

For $f_1, \ldots, f_\ell \in \mathbb{K}[X]$,
let $g = \gcd(f_1, \ldots, f_\ell)$.
Then
$\langle x^{\deg(g)} \rangle = \langle \{ x^{\deg f} \mid f \in I \} \rangle$.

$\Longrightarrow$ **admissible monomial ordering**

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Definition of grevlex

For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$ and $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)$, $\mathbf{x}^{\boldsymbol{\alpha}} \prec_{grlex} \mathbf{x}^{\boldsymbol{\beta}}$ if and only if

- $\deg(\mathbf{x}^{\boldsymbol{\beta}}) > \deg(\mathbf{x}^{\boldsymbol{\alpha}})$,
- $\deg(\mathbf{x}^{\boldsymbol{\beta}}) = \deg(\mathbf{x}^{\boldsymbol{\alpha}})$ and the rightmost nonzero entry of $\boldsymbol{\beta} - \boldsymbol{\alpha} \in \mathbb{Z}^n$ is negative.

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$f_1 = x^2 + y - 1$ and $f_2 = xy + 2$, we compute:

$$S = yf_1 - xf_2 = y^2 - y - 2x$$

Gröbner bases

Definition

Let $\succ$ be an admissible monomial ordering and $G = \{g_1, \ldots, g_p\}$ a set of polynomials of $\mathbb{K}[x_1, \ldots, x_n]$ that generates the ideal I. The set G is a Gröbner basis of $(I, \succ)$ if:

$$\langle \mathrm{lm}_{\succ}(G) \rangle = \langle \mathrm{lm}_{\succ}(I) \rangle.$$

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Elimination Theorem

Let G be a lexicographic Gröbner basis of an ideal I in $\mathbb{K}[x_1, \ldots, x_n]$ with $x_1 \succ \ldots \succ x_n$. Then

$$G \cap \mathbb{K}[x_\ell, \ldots, x_n] \text{ is a Gröbner basis of } I \cap \mathbb{K}[x_\ell, \ldots, x_n]$$

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Worst case: [Mayr and Meyer, 1982]

Gröbner bases and linear algebra

Macaulay Matrices [Macaulay 1902]

$$\begin{pmatrix} \vdots \\ \mathrm{vect}(\boldsymbol{x}^{\boldsymbol{\alpha_i}} f_i) \\ \vdots \end{pmatrix} \xrightarrow[\text{form}]{\text{row echelon}} \text{Gröbner basis truncated to degree } D$$

for $\boldsymbol{\alpha_i}$ with $|\boldsymbol{\alpha_i}| \leq D - \delta$,
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Macaulay bound [Lazard, 1983]

If the sequence $(f_1, \ldots, f_n)$ in $\mathbb{K}[x_1, \ldots, x_n]$ is **regular**, a truncated Gröbner basis to degree $D = 1 + n(d-1)$ for the grevlex ordering is a Gröbner basis of $(\langle f_1, \ldots, f_\ell \rangle, \succ_{\mathrm{grlex}})$.

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Set $\langle f_1, \ldots, f_\ell \rangle = I$ an ideal of $\mathcal{R} = \mathbb{K}[x_1, ..x_n]$,

$$\mathrm{HS}_I(z) = \sum_{d=0} \dim(\mathcal{R}_d / I) z^d \quad \longrightarrow \quad \left(\sum_{i=0}^{\delta - 1} z^i \right)^n$$

F4 Algorithm

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- **S-polynomial** [Buchberger 1965]

For polynomials f_1, f_2,

$$S(f_1, f_2) = \frac{\mathrm{LCM}(\mathrm{lm}_{\succ}(f_1), \mathrm{lm}_{\succ}(f_2))}{\mathrm{lm}_{\succ}(f_1)} f_1 - \frac{\mathrm{LCM}(\mathrm{lm}_{\succ}(f_1), \mathrm{lm}_{\succ}(f_2))}{\mathrm{lm}_{\succ}(f_2)} f_2$$

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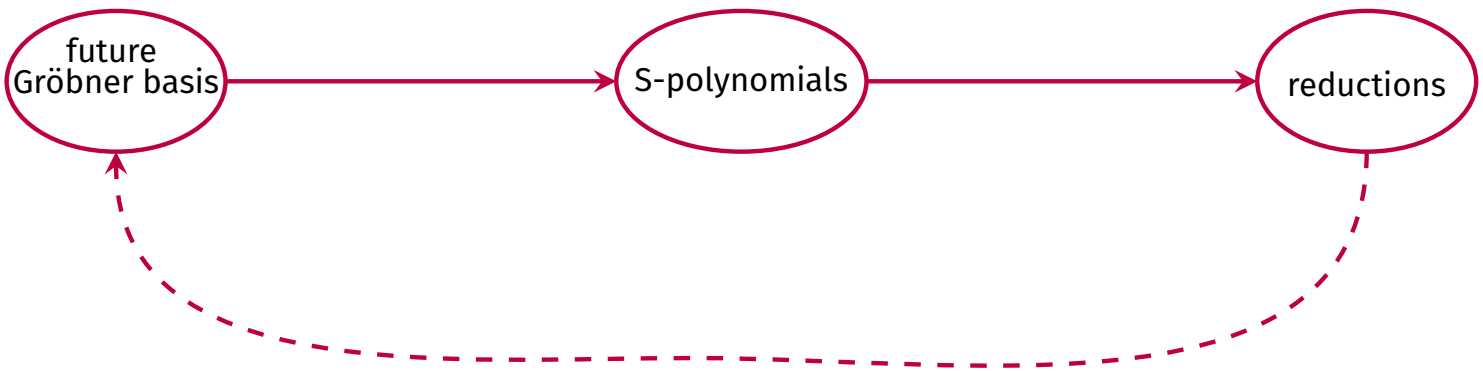

F4 exemple

In $\mathbb{Z}/7\mathbb{Z}[x, y, z]$,

$$f_1 = x^2 + 2xy + 5y^2 + xz + 6yz + 3z^2,$$

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\begin{array}{r} S_{f_1,f_2} \\ S_{f_2,f_3} \end{array} & \left(\begin{array}{cccccccccc} 0 & 0 & 6 & 5 & 3 & 6 & 6 & 6 & 3 & 0 \\ 0 & 0 & 0 & 3 & 3 & 2 & 2 & 4 & 1 & 0 \\ \hline & & & & & & & & & \\ & & & & & & & & & \\ & & & & & & & & & \end{array}\right)
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	x^3	x^2y	xy^2	y^3	x^2z	xyz	y^2z	xz^2	yz^2	z^3
S_{f_1,f_2}	0	0	6	5	3	6	6	6	3	0
S_{f_2,f_3}	0	0	0	3	3	2	2	4	1	0
xf_3	0	0	1	0	4	2	0	3	0	0
yf_3	0	0	0	1	0	4	2	0	3	0
zf_1	0	0	0	0	1	2	5	1	6	3
zf_2	0	0	0	0	0	1	3	4	2	1

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\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 2 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 5 & 0 & 6 \\
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In Software [msolve 2021]**:** F4 is **traced** [Traverso 1989]

1. F4 over $\mathbb{Q} \Longrightarrow$ F4 over $\mathbb{Z}/p_i\mathbb{Z}$ for all $(p_i)_{i\in\{1,\ldots,r\}}$ prime integers
2. For $p_2, \ldots, p_r$**,** re-use information from the computation for p_1

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Upper triangular

Reductors

S-polynomials

$$\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) \Longrightarrow \left(\begin{array}{c|c} Id & A^{-1}B \\ \hline 0 & M \end{array}\right)$$

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- Echelon form of M
- Reductions to zero

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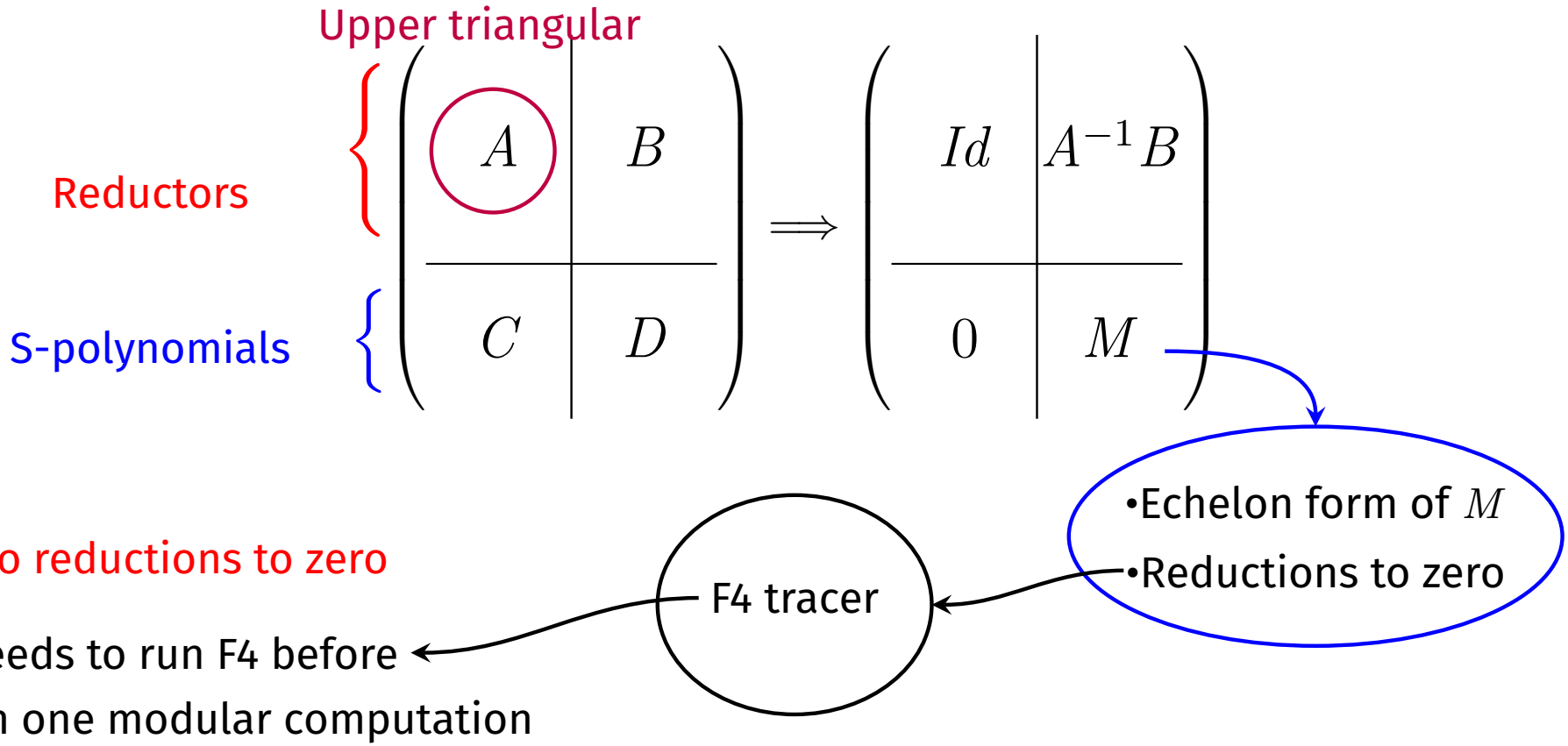

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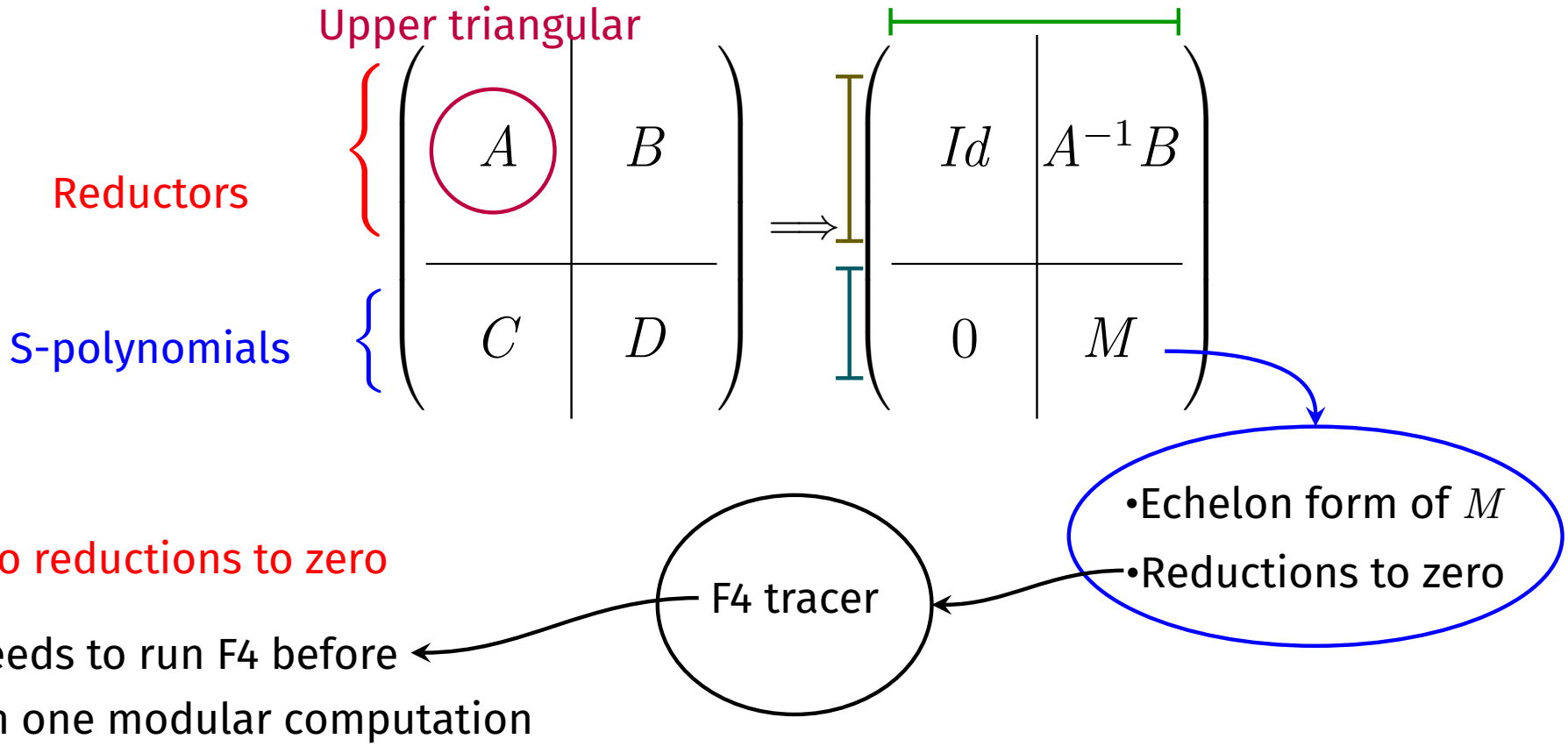

Complexity: $\mathcal{O}\left(\sharp(\text{new GB elements}) \times \sharp(\text{reductors}) \times \sharp(\text{support})\right)$

F5 Algorithm

A new signature system $\longrightarrow$

- No reductions to zero in the regular case
- Must compute Gröbner basis of $\langle f_1, \ldots, f_i \rangle$
- No fast linear algebra

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[Bardet, Faugère, Salvy, 2015]

Under generic assumptions, computing a grevlex Gröbner basis of $\langle f_1, \ldots, f_n \rangle \subset \mathbb{K}[x_1, \ldots x_n]$ takes

$$\underset{n \to +\infty}{\widetilde{\mathcal{O}}} \left(\delta^{3n} 3^n \right)$$

arithmetic operations in $\mathbb{K}$.

Conclusion

- Multivariate polynomial systems can encode a wide variety of situations.
- Gröbner bases are very helpful for eliminating variables and finding solutions to zero-dimensional systems.
- $F4/F5$ is a family of algorithms that compute Gröbner bases for which efficient implementations exist and have proven effective [msolve 2021], [GBLA 2016].

The work in my thesis:

- Understand the complexity of such algorithms in order to improve them.
- Use univariate polynomial matrix transformations (Popov forms) as a generalisation of echelon form to obtain new versions of these algorithms.